The Pythagorean Triplet Generator

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Pythagoras theorem is the one of the first few theorems taught in schools. It is fascinating to many that a triangle with sides 3, 4 and 5 will form a right angle triangle. Other than similar triangles with sides 3, 4 and 5, what other triangles are there with sides of integer values? Triangles with sides of integer values will form a Pythagorean triplet. In particular, a triangle of sides 3, 4 and 5 will be written as (3,4,5) or (4,3,5). We aim to find a 1-parameter Pythagorean triplet generator in the form $(f(t), g(t), h(t))$ such that both $f(t)$ and $g(t)$ are polynomials and $[f(t)]^2 + [g(t)]^2 = [h(t)]^2$ for rational values of t. On top of that we shall also prove the existence of a Pythagorean triplet generator that can generate all Pythagorean triplets.

We first derive a 2-parameter Pythagorean triplet generator, which we then reduce to a 1-parameter generator and which is linked to the t-formula in trigonometry.

By using the formula $(a \pm b)^2 = a^2 \pm 2ab + b^2$and replacing a by u^2, b by v^2, we can easily show that

$$(u^2 - v^2)^2 + (2uv)^2 = (u^2 + v^2)^2$$

Hence, it follows that $(u^2 - v^2, 2uv, u^2 + v^2)$ is a Pythagorean triplet generator. In order to ensure that the three sides of the right-angled triangle are positive, in addition to both u and v being positive integers, the additional condition $u > v$ is needed.

As the right angle triangle with sides $(u^2 - v^2, 2uv, u^2 + v^2)$ is similar to a scenario where all the three lengths are divided by u^2, the Pythagorean triplet can be simplified to $\left(1 - \left(\frac{v}{u}\right)^2, 2\frac{v}{u}, 1 + \left(\frac{v}{u}\right)^2\right)$.

By substituting $= \frac{v}{u}$, the 1-parameter Pythagorean triplet $(1 - t^2, 2t, 1 + t^2)$ is derived.

This should be familiar to many. Recalling when the t-formula is used, substituting $t = \tan\theta$ results in $\sin 2\theta = \frac{2t}{1+t^2}$ and $\cos 2\theta = \frac{1-t^2}{1+t^2}$. Hence, the sides of the right-angled triangle with an acute angle 2θ will coincide with the 1-parameter Pythagorean triplet $(1 - t^2, 2t, 1 + t^2)$.

Thus far, we should bear in mind that t is a positive rational number. To ensure that the three sides are positive quantities, we only need to consider the case when $0 < t < 1$.

Now that a 1-parameter Pythagorean triplet has been successfully generated, this raises the question: Is it possible that other Pythagorean triplet generators exist? Since the 1-parameter Pythagorean triplet generator can be obtained from the t-formula by considering a right-angled triangle with an acute angle θ, intuitively this should extend to cases of general right-angled triangles with an acute angle of $n\theta$ (where $n = 3,4,5, ..$).

In that case, it seems plausible that an 'ultimate' Pythagorean Triplet generator that can generate all possible Pythagorean triplets exists.

In this article, we make an attempt to answer that question. To this end, we investigate the 1-parameter Pythagorean triplet generator $(1-t^2, 2t, 1+t^2)$ in greater detail first.

In the table below, only the values of $= \frac{v}{u}$, where v and u are both positive integers and coprime such that $v < u$ are considered. If these values are not coprime, the fraction can still be reduced to a case where the numerator and denominator are coprime for example, $t = \frac{2}{6} = \frac{1}{3}$.

Using the Pythagorean Triplet generator $(2t, 1-t^2, 1+t^2)$ or $(1-t^2, 2t, 1+t^2)$, Pythagorean triplets as shown in Table 1 can be generated:

t	Pythagorean Triplets (P.T)	Primitive P.T	Remarks	t	Pythagorean Triplets (P.T)	Primitive P.T	Remarks
$1/2$	$(1, 3/4, 5/4)$	(3,4,5)		$1/9$	$(2/9, 80/81, 82/81)$	(9,40,41)	$t = 4/5$
$1/3$	$(2/3, 8/9, 10/9)$	(3,4,5)	$t = 1/2$	$2/9$	$(4/9, 77/81, 85/81)$	(36,77,85)	
$2/3$	$(4/3, 5/9, 13/9)$	(5,12,13)		$4/9$	$(8/9, 65/81, 97/81)$	(65,72,97)	
$1/4$	$(2/4, 15/16, 17/9)$	(8,15,17)		$5/9$	$(10/9, 56/81, 106/81)$	(36,77,85)	$t = 2/7$
$3/4$	$(6/4, 7/16, 25/16)$	(7,24,25)		$7/9$	$(14/9, 32/81, 130/81)$	(16,763,65)	$t = 1/8$
$1/5$	$(2/5, 24/25, 26/25)$	(5,12,13)	$t = 2/3$	$8/9$	$(16/9, 17/81, 145/81)$	(17,144,145)	
$2/5$	$(4/5, 21/25, 29/25)$	(20,21,29)		$1/10$	$(2/10, 99/100, 101/100)$	(20,99,101)	
$3/5$	$(6/5, 16/25, 34/25)$	(8,15,17)	$t = 1/4$	$3/10$	$(6/10, 91/100, 109/100)$	(60,91,109)	
$4/5$	$(8/5, 9/25, 41/25)$	(9,40,41)		$7/10$	$(14/10, 51/100, 149/100)$	(51,140,149)	
$1/6$	$(2/6, 35/36, 37/36)$	(12,35,37)		$9/10$	$(18/10, 19/100, 181/100)$	(19,180,181)	
$5/6$	$(10/6, 11/36, 61/36)$	(11,60,61)		$1/11$	$(2/11, 120/121, 122/121)$	(11,60,61)	$t = 5/6$
$1/7$	$(2/7, 48/49, 50/49)$	(7,24,25)	$t = 3/4$	$2/11$	$(4/11, 117/121, 125/121)$	(44,117,125)	
$2/7$	$(4/7, 45/49, 53/49)$	(28,45,53)		$3/11$	$(6/11, 112/121, 130/121)$	(33,56,65)	$t = 4/7$
$3/7$	$(6/7, 40/49, 58/49)$	(20,21,29)	$t = 2/5$	$4/11$	$(8/11, 105/121, 137/121)$	(88,105,137)	
$4/7$	$(8/7, 33/49, 65/49)$	(33,56,65)		$5/11$	$(10/11, 96/121, 146/121)$	(48,55,73)	$t = 3/8$
$5/7$	$(10/7, 24/49, 74/49)$	(12,35,37)	$t = 1/6$	$6/11$	$(12/11, 85/121, 157/121)$	(85,132,157)	
$6/7$	$(12/7, 13/49, 85/49)$	(13,84,85)		$7/11$	$(14/11, 72/121, 170/121)$	(36,77,85)	$t = 2/9$
$1/8$	$(2/8, 63/64, 65/64)$	(16,63,65)		$8/11$	$(16/11, 51/121, 185/121)$	(57,176,185)	
$3/8$	$(6/8, 55/64, 73/64)$	(48,55,73)		$9/11$	$(18/11, 40/121, 202/121)$	(20,99,101)	$t = 1/10$
$5/8$	$(10/8, 39/64, 89/64)$	(39,80,89)		$10/11$	$(20/11, 21/121, 221/121)$	(21,220,221)	
$7/8$	$(14/8, 15/64, 113/64)$	(15,112,113)		$1/12$	$(2/12, 143/144, 145/144)$	(24,143,145)	

Table 1. Examples of Pythagorean Triplets using the generator $(2t, 1-t^2, 1+t^2)$.

We first observe that different values of t can lead to the same primitive Pythagorean triplet.

Lemma 1.

The choices $t = \frac{v}{u}$ and $t = \frac{u-v}{u+v}$ generate the same primitive Pythagorean triplet.

In particular, $u = 2$ is the only case where u is an even number because of the aforementioned condition that v and u must be coprime.

Proof of Lemma 1.

We need to show that the 1-parameter Pythagorean triplet generator $(1-t^2, 2t, 1+t^2)$ with $t=\frac{v}{u}$ is equivalent to the 1-parameter Pythagorean triplet generator $(1-t^2, 2t, 1+t^2)$ with $t=\frac{u-v}{u+v}$.

We shall first start with $(2t, 1-t^2, 1+t^2)$ where $t=\frac{v}{u}$ then with $t_1=\frac{u-v}{u+v}$.

$$\left(2\frac{v}{u}, 1-\left(\frac{v}{u}\right)^2, 1+\left(\frac{v}{u}\right)^2\right) \equiv (2vu, u^2-v^2, u^2+v^2) \equiv \left(\frac{2(2vu)}{(u+v)^2}, \frac{2(u^2-v^2)}{(u+v)^2}, \frac{2(u^2+v^2)}{(u+v)^2}\right)$$

$$\equiv \left(\frac{(u+v)^2-(u-v)^2}{(u+v)^2}, \frac{2(u-v)}{u+v}, \frac{(u+v)^2+(u-v)^2}{(u+v)^2}\right) \equiv \left(1-\left(\frac{u-v}{u+v}\right)^2, 2\left(\frac{u-v}{u+v}\right), 1+\left(\frac{u-v}{u+v}\right)^2\right)$$

$$\equiv (1-t_1^2, 2t_1, 1+t_1^2) \text{ i.e. } (2t_1, 1-t_1^2, 1+t_1^2) \text{ with } t_1=\frac{u-v}{u+v}.$$

Hence, we have shown that $t=\frac{v}{u}$ generates the same primitive Pythagorean triplet as $t=\frac{u-v}{u+v}$.

For the sake of brevity we denote the 1-parameter Pythagorean triplet generator by $(2t, 1-t^2, 1+t^2)$ in what follows.

There are in fact other possible 1-parameter Pythagorean triplet generators; for example, those generated by using $\tan 3\theta$ and $\tan 4\theta$.

We first consider $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta} = \frac{3t-t^3}{1-3t^2}$ where $t=\tan\theta$. From Pythagoras' theorem, we find that the hypotenuse of the right angle triangle with sides $3t-t^3$ and $1-3t^2$ has length $\sqrt{(1+t^2)^3}$.

$$[1-3t^2]^2+[3t-t^3]^2 = 1-6t^2+9t^4+9t^2-6t^2+t^6 = 1+3t^2+3t^4+t^6 = \left(\sqrt{(1+t^2)^3}\right)^2$$

Hence, the corresponding Pythagorean triplet generator is $\left(3t-t^3, 1-3t^2, \sqrt{(1+t^2)^3}\right)$. However, the term $\sqrt{(1+3t^2)^3}$ is irrational in general. This indicates that using $\tan 3\theta$ will not result in an ideal Pythagorean triplet generator.

We now try $\tan 4\theta$. From $\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta} = \frac{4t-4t^3}{1-6t^2+t^4}$ and Pythagoras' theorem we find that the hypotenuse of the right angle triangle with sides $4t-4t^3$ and $1-6t^2+t^4$ has length $(1+t^2)^2$.

$$[4t-4t^3]^2+[1-6t^2+t^4]^2 = \cdots = 1+4t^2+6t^4+4t^6+t^6 = [(1+t^2)^2]^2$$

Hence, results in the Pythagorean triplet generator $(4t-4t^3, 1-6t^2+t^4, (1+t^2)^2)$. We denote this new generator by $(4t-4t^3, 1-6t^2+t^4, (1+t^2)^2)$.

t	$(4t-4t^3, 1-6t^2+t^4, (1+t^2)^2)$		Remarks (t values in $(2t, 1-t^2, 1+t^2)$)
	Pythagorean Triplet (P.T)	Primitive P.T	
$1/2$	$\left(3/2, -7/16, 25/16\right)$	(7,24,25)	$3/4$ or $1/7$
$1/3$	$\left(32/27, 28/81, 100/81\right)$	(7,24,25)	$3/4$ or $1/7$
$2/3$	$\left(40/27, -119/81, 169/81\right)$	(119,120,169)	$5/12$ or $7/17$
	⋮		⋮
$1/10$	$\left(99/250, -9401/10000, 10201/10000\right)$	(3960,9401,10201)	$20/99$ or $79/119$

Table 2. Examples of Pythagorean Triplets using the generator $(4t-4t^3, 1-6t^2+t^4, (1+t^2)^2)$.

It turns out that various values of t in $(4t-4t^3, 1-6t^2+t^4, (1+t^2)^2)$ correspond to t values in $(2t, 1-t^2, 1+t^2)$ as well. We now explore whether the reverse holds true, that is, whether every value of t in $(2t, 1-t^2, 1+t^2)$ has a corresponding value of t in $(4t-4t^3, 1-6t^2+t^4, (1+t^2)^2)$.

For instance, (3,4,5) is a well-known primitive Pythagorean triplet, where the corresponding value of t in $(2t, 1-t^2, 1+t^2)$ are $1/2$ and $1/3$. We now find the corresponding value of t in $(4t-4t^3, 1-6t^2+t^4, (1+t^2)^2)$ to generate the Pythagorean triplet (3,4,5).

Essentially, there are 4 cases to consider: $\frac{(1+t^2)^2}{4t-4t^3} = \pm\frac{5}{3} \text{ or } \pm\frac{5}{4}$

Case 1: $\frac{(1+t^2)^2}{4t-4t^3} = \frac{5}{3} \Rightarrow t^4 + \frac{20}{3}t^3 + 3t^2 - \frac{20}{3}t + 1 = 0$

$$(t^2+6t-1)\left(t^2+\frac{5}{3}t-1\right) = 0 \;\; \Rightarrow t = -3-\sqrt{10}, -3+\sqrt{10}, -\frac{3+\sqrt{10}}{3} \text{ or } -\frac{3-\sqrt{10}}{3}$$

Case 2: $\frac{(1+t^2)^2}{4t-4t^3} = -\frac{5}{3} \Rightarrow \cdots \Rightarrow t = 3+\sqrt{10}, 3-\sqrt{10}, \frac{3+\sqrt{10}}{3} \text{ or } \frac{3-\sqrt{10}}{3}$

Case 3: $\frac{(1+t^2)^2}{4t-4t^3} = \frac{5}{4} \Rightarrow \cdots \Rightarrow t = -2-\sqrt{5}, -2+\sqrt{5}, -\frac{1+\sqrt{5}}{2} \text{ or } -\frac{1-\sqrt{5}}{2}$

Case 4: $\frac{(1+t^2)^2}{4t-4t^3} = -\frac{5}{4} \Rightarrow \cdots \Rightarrow t = 2-\sqrt{5}, 2+\sqrt{5}, \frac{1+\sqrt{5}}{2} \text{ or } \frac{1-\sqrt{5}}{2}$

None of these four cases result in rational values of t. Hence, this shows that $(4t-4t^3, 1-6t^2+t^4, (1+t^2)^2)$ is unable to generate all the elements in $(2t, 1-t^2, 1+t^2)$.

This seems to suggest that while $(2t, 1-t^2, 1+t^2)$ is able to generate all the elements in $(4t-4t^3, 1-6t^2+t^4, (1+t^2)^2)$, the reverse is not true.

Next, we show that $(2T, 1-T^2, 1+T^2)$ can generate all the elements in $(4t-4t^3, 1-6t^2+t^4, (1+t^2)^2)$.

Let $T = \frac{2t}{1-t^2}$

$$\begin{aligned}(2T, 1-T^2, 1+T^2) &\equiv \left(2\left(\frac{2t}{1-t^2}\right), 1-\left(\frac{2t}{1-t^2}\right)^2, 1+\left(\frac{2t}{1-t^2}\right)^2\right) \\ &\equiv (2(2t)(1-t^2), (1-t^2)^2-(2t)^2, (1-t^2)^2+(2t)^2) \\ &\equiv (4t-4t^3, 1-6t^2+t^4, (1+t^2)^2)\end{aligned}$$

Hence, the Pythagorean triplets generated by $(4t-4t^3, 1-6t^2+t^4, (1+t^2)^2)$ are all subsets of $(2t, 1-t^2, 1+t^2)$.

This suggests that $(2T, 1-T^2, 1+T^2)$ can generate all existing Pythagorean triplets.

Firstly, let a Pythagorean triplet generator be expressed as $(kt, g(t), f(t))$, where $k \in \mathbb{Q}/\{0\}$. Then,

$$(kt)^2 + [g(t)]^2 = [f(t)]^2 \text{ and}$$

$$(kt)^2 = [f(t)-g(t)][f(t)+g(t)]$$

Based on the following table, there are 3 possibilities when $\deg(kt)^2 = 2$:

Case 1: $\deg[f(t) - g(t)] = 1$ and $\deg[f(t) + g(t)] = 1$

Case 2: $\deg[f(t) - g(t)] = 2$ and $\deg[f(t) + g(t)] = 0$

Case 3: $\deg[f(t) - g(t)] = 0$ and $\deg[f(t) + g(t)] = 2$

<u>Case 1</u>

Let $f(t) - g(t) = \alpha t + \beta$ and $f(t) + g(t) = \alpha_1 t + \beta_1$. Then $2f(t) = (\alpha + \alpha_1)t + (\beta + \beta_1)$

Hence, $\deg\big(f(t)\big) = 1$ and $\deg(g(t)) \leq 1$

Let $f(t) = a_1 t + b_1, g(t) = a_2 t + b_2$ and note that $g(t) = a_2 t + b_2$ covers both cases of $\deg(g(t)) \leq 1$

If $\deg(g(t)) = 1$, then $g(t) = a_2 t + b_2$ and

$$k^2 t^2 \equiv [(a_1 - a_2)t + b_1 - b_2][(a_1 + a_2)t + b_1 + b_2]$$

Compare the constant term:

$$0 = {b_1}^2 - {b_2}^2 \text{ i.e. } b_1 = \pm b_2$$

Comparing the coefficient of t:

$$(b_1 - b_2)(a_1 + a_2) + (a_1 - a_2)(b_1 + b_2) = 0 \qquad (1)$$

If $b_1 = b_2$, then (1) will become $(a_1 - a_2)(b_1 + b_2) = 0$ resulting $a_1 = a_2$ since $b_1 + b_2 \neq 0$.

If $a_1 = a_2$ and $b_1 = b_2$, then $f(t) = g(t) \Longrightarrow k = 0$ (not applicable)

If $b_1 = -b_2$, then (1) will become $(b_1 - b_2)(a_1 + a_2)0$ resulting $a_1 = -a_2$ since $b_1 - b_2 \neq 0$.

If $a_1 = -a_2$ and $b_1 = -b_2$, then $f(t) = -g(t) \Longrightarrow k = 0$ (not applicable)

$\therefore$ Case 1 is not possible.

Since Cases 2 and 3 are similar, we only consider Case 3

Let $f(t) - g(t) = c_1$ and $f(t) + g(t) = a_0 + b_0 t + c_0 t^2$

Hence, $f(t) = \frac{1}{2}(a_0 + c_1 + b_0 t + c_0 t^2)$ and $g(t) = \frac{1}{2}(a_0 - c_1 + b_0 t + c_0 t^2)$

Let $f(t) = \gamma_0 + \beta_0 t + \alpha_0 t^2$ and

$$g(t) = \gamma_1 + \beta_1 t + \alpha_1 t^2$$

Since $\deg[f(t) + g(t)] = 2$ and $\deg[f(t) - g(t)] = 0$

$\therefore \alpha_0 = \alpha_1$ and $\beta_0 = \beta_1$

$$k^2t^2 \equiv (\gamma_0 - \gamma_1)[2\alpha_0 t^2 + 2\beta_0 t + \gamma_0 + \gamma_1]$$

Compare the constant term:

$$0 = {\gamma_0}^2 - {\gamma_1}^2 \text{ i.e. } \gamma_0 = \pm\gamma_1$$

If $\gamma_0 = \gamma_1$, then $f(t) = g(t) \Longrightarrow k = 0$ (not applicable)

If $\gamma_0 = -\gamma_1$,

By comparing the coefficient of t:

$$0 = 2(\gamma_0 - \gamma_1)\,\beta_0 \text{ i.e. } \beta_0 = 0$$

By comparing the coefficient of t^2:

$$k^2 = 2\alpha_0(\gamma_0 - \gamma_1) = 4\alpha_0\gamma_0$$

$$k = 2\sqrt{\alpha_0\gamma_0} \text{ or } k = -2\sqrt{\alpha_0\gamma_0}$$

Without loss of generality, we just need to consider $k = 2\sqrt{\alpha_0\gamma_0}$

Recalling that $\gamma_0 = -\gamma_1$, $\alpha_0 = \alpha_1$ and $\beta_0 = \beta_1 = 0$, thus

$f(t) = \alpha_0 t^2 + \gamma_0$ and $g(t) = \alpha_0 t^2 - \gamma_0$

Thus we have expressed the Pythagorean triplet generator as $(kt, g(t), f(t))$, where $k = 2\sqrt{\alpha_0\gamma_0}$, $f(t) = \alpha_0 t^2 + \gamma_0$ and $g(t) = \alpha_0 t^2 - \gamma_0$.

Hence, the Pythagorean triplet generator is $\left(2\sqrt{\alpha_0\gamma_0}t, \alpha_0 t^2 - \gamma_0, \alpha_0 t^2 + \gamma_0\right)$ and by factoring out γ_0, the Pythagorean triplet generator becomes $\left(2\sqrt{\frac{\alpha_0}{\gamma_0}}t, \left(\frac{\alpha_0}{\gamma_0}\right)t^2 - 1, \left(\frac{\alpha_0}{\gamma_0}\right)t^2 + 1\right)$. However, if $= \sqrt{\frac{\alpha_0}{\gamma_0}}t$, then it can be expressed as $(2T, T^2 - 1, T^2 + 1)$, which is equivalent to $(2T, 1 - T^2, 1 + T^2)$.

Hence, $(2T, 1 - T^2, 1 + T^2)$ is the ultimate 1-parameter Pythagorean triplet generator that will generate all existing Pythagorean triplets.