The General Pascal's Triangle

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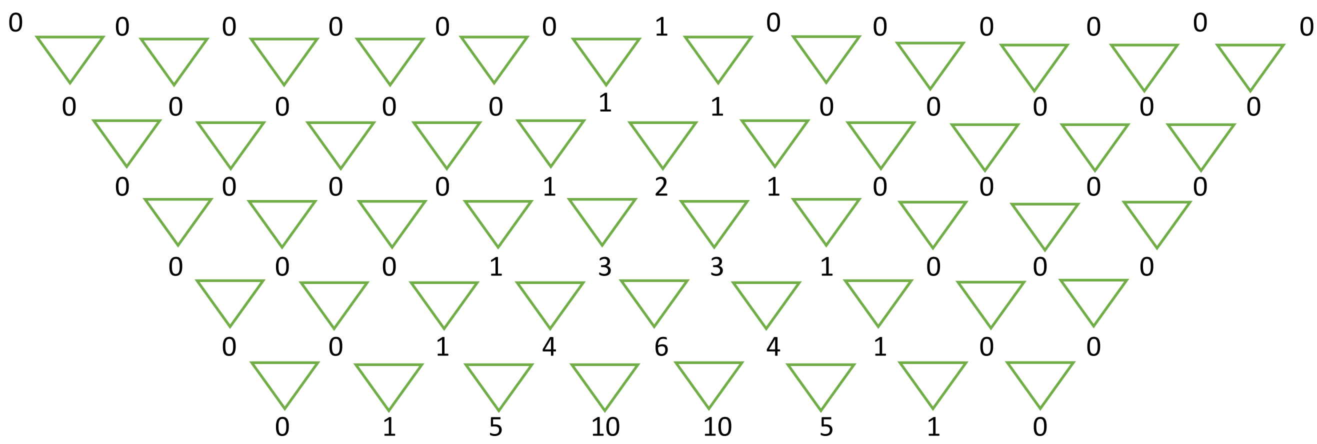
This article aims to firstly illustrate how Binomial coefficients coincide with Pascal's triangle, and secondly show that the Fibonacci sequence is embedded inside Pascal's triangle. Additionally, this article features a general form of Pascal's Triangle and proved that a general form of a second order difference equation is also embedded inside the General Pascal's Triangle.

To first understand how Pascal's triangle works, it is important to note that the entire Pascal's triangle can be generated with just the number "1". Based on the following figure, there is a number "1" in the middle and infinite zeroes that extend past both sides of "1" (left and right). By adding two adjacent terms, another term is created in the following row (as represented by the numerous inverted triangles). This is how Pascal's triangle is generated continuously.



Next, this article aims to prove how the numbers in Pascal's triangle coincide with Binomial coefficients.

However, first and foremost, the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ is essential in proving this relationship. The proof for this formula can be argued using combinatorics.

Consider a scenario where there are r identical A letters and $(n-r)$ identical B letters to be arranged in a straight line.

This problem can be solved by one of the 3 following methods:

Method 1: Out of the n slots chosen, there are r slots to place the A letters. As for the rest of the $(n-r)$ slots, there is only 1 choice to allocate the rest of the B letters. Hence, this scenario can be solved in $\binom{n}{r}$ ways.

Method 2: By solving this directly, there are $\frac{n!}{r!(n-r)!}$ ways to rearrange all the letters.

Method 3: Out of the slots n, $(n-r)$ slots are chosen to place the B letters. As for the rest of the r slots, there is only 1 choice to allocate the rest of the A letters. Hence, this scenario can be solved in $\binom{n}{n-r}$ ways.

Since all 3 methods are solving the same scenario, their answers are all equivalent.

Hence, $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \binom{n}{n-r}$.

Next, $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ needs to be used to explain $0! = 1$ because the first number "1" in the first row of Pascal's triangle is $\binom{0}{0} = \frac{0!}{0!(0-0)!} = 1$.

By substituting $r = n$, the result is $\binom{n}{n} = 1$. Hence, $\binom{n}{n} = \frac{n!}{n!(n-n)!} = 1$ and $0! = 1$

By observation, since $\binom{n}{k} = 0$ for $k < 0$ or $k > n$, where $n \in \mathbb{Z}^+ \cup \{0\}$ and $k \in \mathbb{Z}$, this can be used to explain the infinite series of zeroes that extend past the left and right borders of Pascal's Triangle.

Finally, the formula $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$ needs to proven. Since proof using algebraic manipulation is commonly known, this article has created a new method of proving by using a combinatorics argument.

The proof begins by choosing from n ladies and 1 gentleman to form a committee of $r + 1$ people. The number of ways can be calculated using two methods.

Method 1: The direct method is simply $\binom{n+1}{r+1}$ ways.

Method 2: There are 2 cases to be considered, namely whether the gentleman is chosen or not.

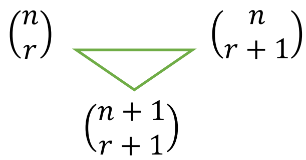
Case 1: If the gentleman is chosen, there is a need to choose another r ladies from the n ladies, which results in $\binom{n}{r}$ ways.

Case 2: If the gentleman is not chosen, there is a need choose another $r + 1$ ladies from the n ladies, which results in $\binom{n}{r+1}$ ways.

Note that case 1 and 2 are mutually exclusive, hence, the total number of ways is $\binom{n}{r} + \binom{n}{r+1}$.

Since both methods are solving the same question, hence, $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$.

This can be rearranged to express the following geometrically:



Based on the figure above, it is now clear that the numbers in Pascal's triangle and binomial coefficients must coincide.

Next, this article shall explain and prove the observation that the Fibonacci sequence is also embedded inside Pascal's triangle.

Recalling the Fibonacci sequence:

$$u_{n+2} = u_{n+1} + u_n \, ; \, u_1 = u_2 = 1, n \in \mathbb{Z}^+$$

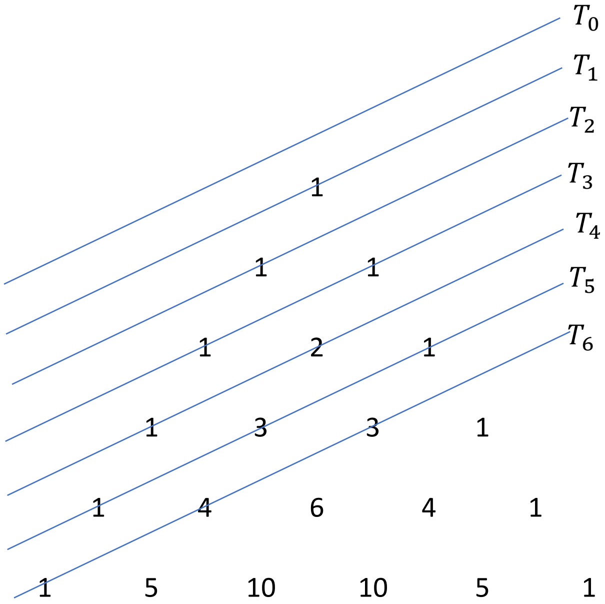
This can be extended to

$$v_{n+2} = v_{n+1} + v_n \, ; \, v_0 = 0, v_1 = 1, n \in \mathbb{Z}^+ \cup \{0\}$$

As such, v_n is defined as follows:

$$v_n = \begin{cases} 0 & \text{for} \quad n = 0 \\ u_n & \text{for} \quad n \in \mathbb{Z}^+ \end{cases}$$

By relooking at Pascal's triangle in the following manner:



$T_0 = 0$

$T_1 = 1$

$T_2 = 1$

$T_3 = 1 + 1 = 2$

$T_4 = 1 + 2 = 3 = T_2 + T_3$

$T_5 = 1 + 3 + 1 = 5 = T_3 + T_4$

$T_6 = 1 + 4 + 3 = 8 = T_4 + T_5$

As such, it can be observed that T_n needs to be considered when $n = 2m$ and $n = 2m + 1$ where $m \geq 1$ with $T_0 = 0$:

Case 1: When $n = 2m$,

$$T_n = T_{2m} = \binom{2m-1}{0} + \binom{2m-2}{1} + \binom{2m-3}{2} + \cdots + \binom{2m-m}{m-1}$$

$$T_{n-1} = T_{2m-1} = \binom{2m-2}{0} + \binom{2m-3}{1} + \binom{2m-4}{2} + \cdots + \binom{m-1}{m-1}$$

$$T_{n-2} = T_{2m-2} = \binom{2m-3}{0} + \binom{2m-4}{1} + \binom{2m-5}{2} + \cdots + \binom{m-1}{m-2}$$

$$T_{n-1} + T_{n-2} = \binom{2m-2}{0} + \left[\binom{2m-3}{0} + \binom{2m-3}{1}\right] + \left[\binom{2m-4}{1} + \binom{2m-4}{2}\right] + \cdots + \left[\binom{m-1}{m-2} + \binom{m-1}{m-1}\right]$$

$$= \binom{2m-1}{0} + \binom{2m-2}{1} + \binom{2m-3}{2} + \cdots + \binom{m}{m-1} = T_{2m} = T_n \qquad \text{(shown)}$$

Note that the formula $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$ was applied repeatedly.

Case 2: When $n = 2m + 1$,

$$T_n = T_{2m+1} = \binom{2m}{0} + \binom{2m-1}{1} + \binom{2m-2}{2} + \cdots + \binom{2m-m}{m}$$

$$T_{n-1} = T_{2m} = \binom{2m-1}{0} + \binom{2m-2}{1} + \binom{2m-3}{2} + \cdots + \binom{m}{m-1}$$

$$T_{n-2} = T_{2m-1} = \binom{2m-2}{0} + \binom{2m-3}{1} + \binom{2m-4}{2} + \cdots + \binom{m-1}{m-1}$$

$$T_{n-1} + T_{n-2} = \binom{2m-1}{0} + \left[\binom{2m-2}{0} + \binom{2m-2}{1}\right] + \left[\binom{2m-3}{1} + \binom{2m-3}{2}\right] + \cdots + \left[\binom{m-1}{m-1} + \binom{m}{m-1}\right]$$

$$= \binom{2m}{0} + \binom{2m-1}{1} + \binom{2m-2}{2} + \cdots + \binom{m}{m} = T_{2m+1} = T_n \quad \text{(shown)}$$

Hence, it has been proven successfully that the Fibonacci sequence is embedded inside Pascal's triangle by using Binomial coefficients.

Next, this article will consider the second order difference equation $u_{n+2} = \alpha u_{n+1} + \beta u_n$; $u_0 = 0, u_1 = 1$, $n \in \mathbb{Z}^+ \cup \{0\}$ such that both α and β are some constants.

As such, it will generate $u_0 = 0, u_1 = 1, u_2 = \alpha, u_3 = \alpha^2 + \beta, u_4 = \alpha(\alpha^2 + \beta) + \beta(\alpha) = \alpha^3 + 2\alpha\beta$,

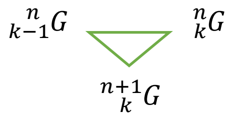
$u_5 = \alpha(\alpha^3 + 2\alpha\beta) + \beta(\alpha^2 + \beta) = \alpha^4 + 3\alpha^2\beta + \beta^2, u_6 = \alpha(\alpha^4 + 3\alpha^2\beta + \beta^2) + \beta(\alpha^3 + 2\alpha\beta)$

i.e. $u_6 = \alpha^5 + 4\alpha^3\beta + 3\alpha\beta^2$, and so on…

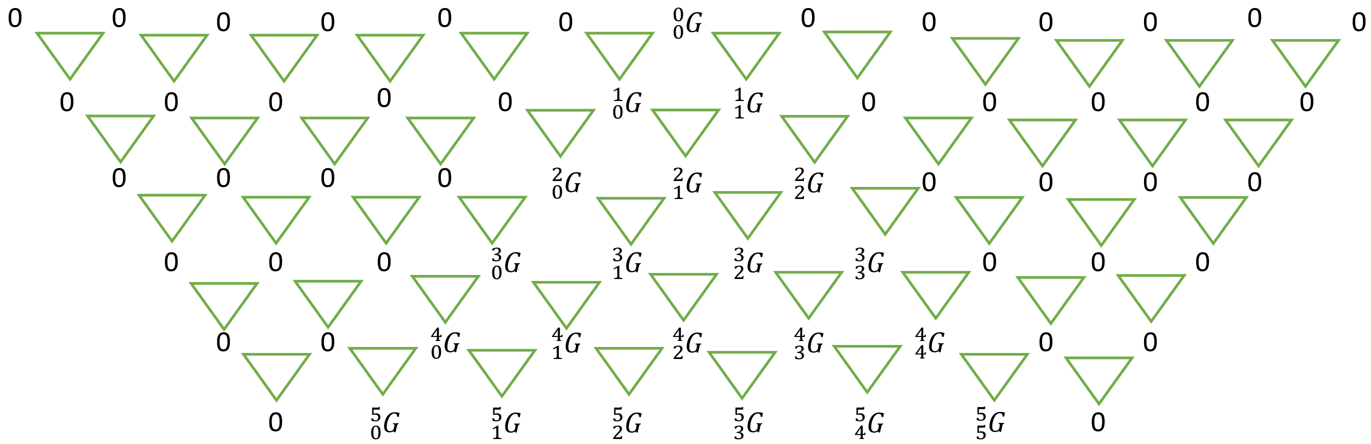
<u>The General Pascal's Triangle</u>

We define ${}^{n}_{k}G = 0$ for $k < 0$ or $k > n$ where $n \in \mathbb{Z}^+ \cup \{0\}$, $k \in \mathbb{Z}$ with ${}^{n}_{k}G = 0$ for $n \in \mathbb{Z}^-$ and ${}^{0}_{0}G = 1$.

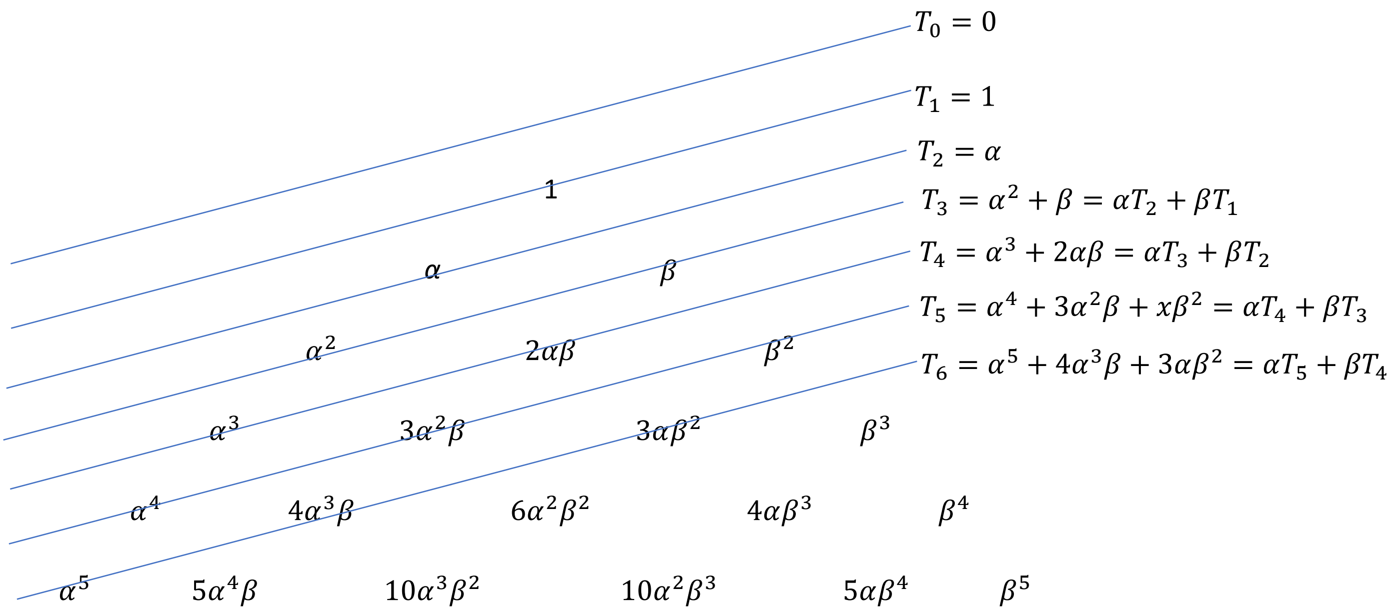
We also define $\beta({}^{\;\,n}_{k-1}G) + \alpha({}^{n}_{k}G) = {}^{n+1}_{\;\,k}G$. This can be rearrange this to express the following geometrically:



As such, the following General Pascal's Triangle can be successfully generated:



By relooking at the General Pascal's triangle in the following manner:



Similarly, one can observe that T_n needs to be considered when $n = 2m$ and $n = 2m + 1$, where $m \geq 1$ with $T_0 = 0$:

Case 1: When $n = 2m$, $T_n = T_{2m} = {}^{2m-1}_{0}G + {}^{2m-2}_{1}G + {}^{2m-3}_{2}G + \cdots + {}^{2m-m}_{m-1}G$

$$T_{n-1} = T_{2m-1} = {}^{2m-2}_{0}G + {}^{2m-3}_{1}G + {}^{2m-4}_{2}G + \cdots + {}^{m-1}_{m-1}G$$

$$T_{n-2} = T_{2m-2} = {}^{2m-3}_{0}G + {}^{2m-4}_{1}G + {}^{2m-5}_{2}G + \cdots + {}^{m-1}_{m-2}G$$

$\alpha T_{n-1} + \beta T_{n-2}$

$= \alpha({}^{2m-2}_{0}G) + [\beta({}^{2m-3}_{0}G) + \alpha({}^{2m-3}_{1}G)] + [\beta({}^{2m-4}_{1}G) + \alpha({}^{2m-4}_{2}G)] + \cdots + [\beta({}^{m-1}_{m-2}G) + \alpha({}^{m-1}_{m-1}G)]$

$= {}^{2m-1}_{0}G + {}^{2m-2}_{1}G + {}^{2m-3}_{2}G + \cdots + {}^{2m}_{m-1}G = T_{2m} = T_n$ (shown)

Note that we apply the formula $\beta({}^{n}_{k-1}G) + \alpha({}^{n}_{k}G) = {}^{n+1}_{k}G$ repeatedly many times.

Case 2: When $n = 2m + 1$, $T_n = T_{2m+1} = {}^{2m}_{0}G + {}^{2m-1}_{1}G + {}^{2m-2}_{2}G + \cdots + {}^{2m-m}_{m}G = T_{2m+1}$

$$T_{n-1} = T_{2m} = {}^{2m-1}_{0}G + {}^{2m-2}_{1}G + {}^{2m-3}_{2}G + \cdots + {}^{m}_{m-1}G$$

$$T_{n-2} = T_{2m-1} = {}^{2m-2}_{0}G + {}^{2m-3}_{1}G + {}^{2m-4}_{2}G + \cdots + {}^{m-1}_{m-1}G$$

$\alpha T_{n-1} + \beta T_{n-2}$

$= \alpha({}^{2m-1}_{0}G) + [\beta({}^{2m-2}_{0}G) + \alpha({}^{2m-1}_{1}G)] + [\beta({}^{2m-3}_{1}G) + \alpha({}^{2m-3}_{2}G)] + \cdots + [\beta({}^{m-1}_{m-1}G) + \alpha({}^{2m-m}_{m}G)]$

$= {}^{2m}_{0}G + {}^{2m-1}_{1}G + {}^{2m-2}_{2}G + \cdots + {}^{2m-m}_{m}G = T_{2m+1} = T_n$ (shown)

Hence, it has been proven successfully that the sequence of a general 2nd order difference equation, with initial conditions $u_0 = 0, u_1 = 1$ and $u_{n+2} = \alpha u_{n+1} + \beta u_n$, $n \in \mathbb{Z}^+ \cup \{0\}$ is embedded inside the General Pascal's Triangle.

In the General Pascal's triangle, it is obvious that the terms that appear in every horizontal row are the expansion of $(\alpha + \beta)^n$ whereby $n \in \mathbb{Z}^+ \cup \{0\}$. It is also interesting to note that α and β are assumed to be constants which are not restricted to only real numbers. In fact, α and β can also be complex numbers.

The trinomial coefficients can be represented in a 3-dimensional tetrahedron diagram. Furthermore, it may be possible to show geometrically how a general 3rd order difference equation with specific conditions is embedded inside the 3-dimensional tetrahedron diagram. Perhaps, this shall be explored in future publications.