In teaching complex numbers, i^i is a classic example most teachers give with the following solution:

$$i = 1e^{i\left(\frac{\pi}{2}\right)} \Longrightarrow i^i = \left[e^{i\left(\frac{\pi}{2}\right)}\right]^i = e^{-\frac{\pi}{2}} \text{ i.e. } i^i \text{ is a real number.}$$

However, when I read from a book many years ago that actually i^i has infinitely many answers I was pleasantly surprised and thought of the following solution instead:

$i^i = \left[e^{i\left(\frac{\pi}{2}+2k\pi\right)}\right]^i = e^{-\left(\frac{\pi}{2}+2k\pi\right)}$ where $k \in \mathbb{Z}$ i.e. i^i has infinitely many answers and all are real numbers.

As such I set the following questions to my students:

Find the modulus and principle argument of the following:

a) e^i; b) 1^i; c) $a^i, a \in \mathbb{R}^+$; d) (-1)^i; e) $a^i, a \in \mathbb{R}^-$; f) 0^i; g) $c^i, c \in \mathbb{C}/\{0\}$

a) $e^i = \left[e.e^{i(2k\pi)}\right]^i$ $= e^{-2k\pi}e^i$ i.e. $\left\|e^{-2k\pi}\right\| = 1$ i.e. $k = 0$	b) $1^i = \left[e^{i(0+2k\pi)}\right]^i$ $= e^{-2k\pi}$ Similar to a), $k = 0$	c) $a^i = \left[a.e^{i(0+2k\pi)}\right]^i$ $= a^i e^{-2k\pi}$ $= e^{-2k\pi}e^{i\ln a}$ Similarly, $k = 0$	d) $(-1)^i = \left(i^i\right)^2$ $= \left[e^{-\left(\frac{\pi}{2}+2k\pi\right)}\right]^2$ $= e^{-(\pi+4k\pi)}$
$\left\|e^i\right\| = 1$ Prin arg $\left(e^i\right)$=1	$\left\|1^i\right\| = 1$ Prin arg $\left(1^i\right)$=0	$\left\|a^i\right\| = 1$ Prin arg $\left(a^i\right) = \ln a$	$\left\|a^i\right\| = e^{-(\pi+4k\pi)}$ Prin arg $\left((-1)^i\right) = 0$

e) $a^i = [-a]^i(-1)^i$ $= e^{i\ln(-a)}e^{-(\pi+4k\pi)}$ $= e^{-\pi(1+4k)}e^{i\ln(-a)}$	f) $0^i = \left[0.e^{i(\theta+2k\pi)}\right]^i$ $= 0^i e^{-(\theta+2k\pi)}$ $0^i\left(1-e^{-(\theta+2k\pi)}\right) = 0$ But this is true $\forall -\pi < \theta \leq \pi$, thus $0^i = 0$	g) Let $c = re^{i(\theta+2k\pi)}$, $r = \|c\| > 0$ and Prin arg $c = \theta \neq 0$ $c^i = \left[r^i e^{-(\theta+2k\pi)}\right]$ $= e^{-(\theta+2k\pi)}e^{i\ln r}$
$\left\|a^i\right\| = e^{-(1+4k)\pi}$ Prin arg $\left(a^i\right) = \ln(-a)$	$\left\|0^i\right\| = 0$ Prin arg $\left(0^i\right)$=undefined	$\left\|c^i\right\| = e^{-(\theta+2k\pi)}$ Prin arg $\left(c^i\right) = \ln r = \ln\|c\|$

In the above working, k is any integer.

Note that for d), e) and g) there are infinitely many answers for the modulus.

Thinking questions:

Q1) What is i^{i^i}? Is it $\left[e^{-\left(\frac{\pi}{2}+2k\pi\right)}\right]^i$ or $i^{e^{-\left(\frac{\pi}{2}+2k\pi\right)}}$?

Q2) If both z and w are complex numbers, how to evaluate z^w?