Heron's Formula via Proofs Without Words

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Heron's remarkable formula, $K = \sqrt{s(s-a)(s-b)(s-c)}$, for the area K of a triangle with side lengths a, b, and c, and *semiperimeter* $s = (a+b+c)/2$, can be proved by a number of methods. Algebraic, geometric, trigonometric and function-theoretic proofs can be found in [**1**], [**2**], [**4**], [**5**], [**6**], [**7**], [**9**], [**10**], [**12**], and [**14**]. The purpose of this Capsule is to use "proofs without words" to establish two lemmas (which are of interest in their own right) that reduce the proof of Heron's formula to elementary algebra. It is based on the proof found in [**2**] (which reappears in [**4**] and [**10**]).

Let $\triangle ABC$ be a triangle with sides a, b, c, as in Figure 1(a), and bisect each angle to locate the center of the *incircle* (as did Heron). Extending an *inradius* (length r) to each side now partitions the triangle into six smaller right triangles, with side lengths as indicated in Figure 1(b).

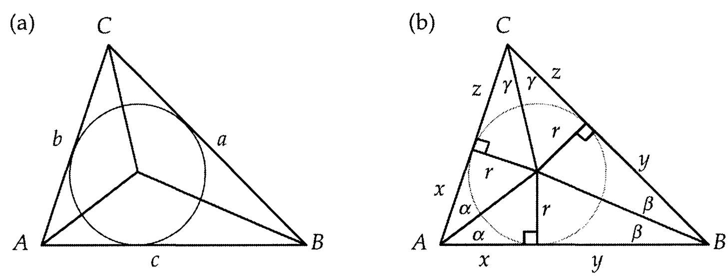

Figure 1.

Note that the semiperimeter satisfies

$$s = x + y + z = x + a = y + b = z + c. \quad (1)$$

We now prove (without words) the two lemmas from which Heron's formula readily follows.

Lemma 1. *The area K of a triangle is equal to the product of its inradius and semiperimeter.*

Proof.

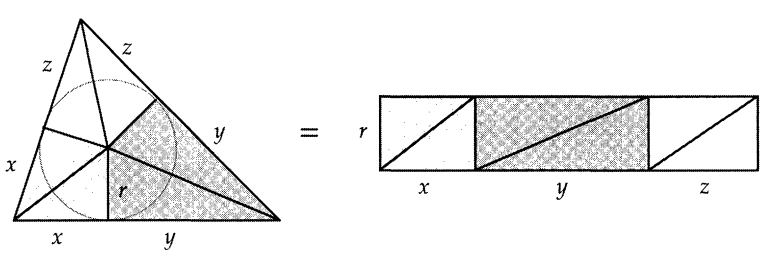

Figure 2. $K = r(x+y+z) = rs$.

For another proof without words of Lemma 1, see [**8**].

Lemma 2. *If α, β and γ are any positive angles such that $\alpha+\beta+\gamma=\pi/2$, then*

$$\tan\alpha\tan\beta+\tan\beta\tan\gamma+\tan\gamma\tan\alpha=1.$$

Proof.

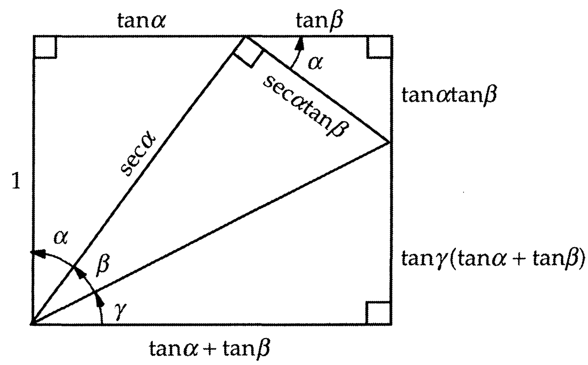

Figure 3. $\tan\alpha\tan\beta+\tan\beta\tan\gamma+\tan\gamma\tan\alpha=1$.

Theorem 1 (Heron's Formula). *The area K of a triangle with sides of length a, b, and c is given by*

$$K=\sqrt{s(s-a)(s-b)(s-c)}$$

where $s=(a+b+c)/2$.

Proof. Applying Lemma 2 to the angles labeled α, β, and γ in Figure 1(b) yields

$$\begin{aligned}1&=\tan\alpha\tan\beta+\tan\beta\tan\gamma+\tan\gamma\tan\alpha,\\&=\frac{r}{x}\cdot\frac{r}{y}+\frac{r}{y}\cdot\frac{r}{z}+\frac{r}{z}\cdot\frac{r}{x},\\&=\frac{r^2(x+y+z)}{xyz}=\frac{r^2s}{xyz}=\frac{K^2}{sxyz},\end{aligned}$$

where the last step follows from Lemma 1. Invoking (1) yields

$$K^2=sxyz=s(s-a)(s-b)(s-c),$$

which completes the proof.

We conclude with a comment and a challenge. A quadrilateral is called *cyclic* if it possesses a circumscribing circle. If the side lengths of a cyclic quadrilateral $ABCD$ are a, b, c, and d, then Brahmagupta's formula for its area is $K=\sqrt{(s-a)(s-b)(s-c)(s-d)}$ [**3, 13**], where $s=(a+b+c+d)/2$ again denotes the semiperimeter (when $d=0$, Brahmagupta's formula reduces to Heron's formula).

If $ABCD$ also has an incircle, then $K = \sqrt{abcd}$ [**11**]. We challenge the reader to find appropriate generalizations of the lemmas in this Capsule (and proofs, perhaps without words) for these expressions.

References

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A Property of Quadrilaterals

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What if you were to draw a quadrilateral that was quite unlike a parallelogram? In an effort to disrupt the parallelism of pairs of opposite sides, you might construct the opposite interior angles to be as different as possible, or, equivalently, have the diagonals most clearly not bisect each other. The diagonals of a quadrilateral play a key role in many associated properties, especially with *cyclic* quadrilaterals, ones whose vertices lie on a circle [**1**]. For instance, Ptolemy's Theorem tells us that the product of the lengths of the diagonals of a cyclic quadrilateral equals the sum of the products of the lengths of the pairs of opposite sides. Cyclic quadrilaterals are also noteworthy because those are the ones of maximum area formed from four given sides. But even for an arbitrary quadrilateral, there is a remarkable relationship (originally proved by Euler) between the diagonals and the four sides. If we are given a convex quadrilateral $ABCD$, as shown in the figure, and a, b, c, d, are the lengths of the four sides, then the sum of the squares of these sides is related to the lengths of the two diagonals by

$$a^2 + b^2 + c^2 + d^2 = \overline{AC}^2 + \overline{BD}^2 + 4x^2,$$

where x is the length of the segment connecting the midpoints of the two diagonals. This $4x^2$ term can conveniently be thought of as measuring how far an ordinary quadrilateral differs from being a parallelogram.

