Tran Minh Tri used a question in tutorial * to first find the radius of the in-circle of a triangle before proving the heron formula.

Heron formula: Area of a triangle $= \sqrt{s(s-a)(s-b)(s-c)}$ where $2s = a + b + c$

Radius of the inscribed circle of a triangle is $r = \sqrt{\dfrac{(s-a)\cdot(s-b)\cdot(s-c)}{s}}$, where s is half the perimeter of the angle.

Side proof:

If $\alpha+\beta+\gamma=180°$, then $\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\cdot\tan\beta\cdot\tan\gamma$

We have, $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\cdot\tan\beta}$

$$\Rightarrow \tan\alpha+\tan\beta=\tan(\alpha+\beta)\cdot(1-\tan\alpha\cdot\tan\beta)$$

When $\alpha+\beta+\gamma=180°$,

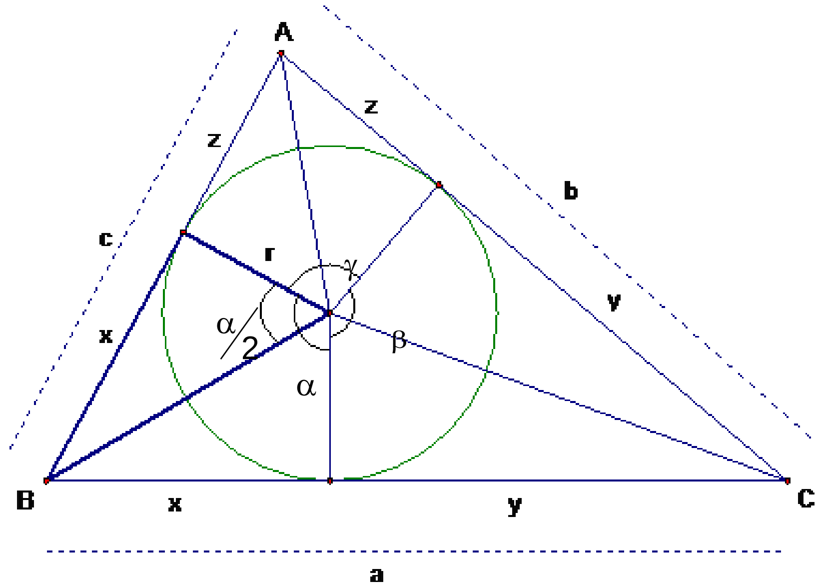
$\Rightarrow$ $\alpha+\beta=180°-\gamma \Leftrightarrow \tan(\alpha+\beta)=-\tan\gamma$

and $\tan(\alpha+\beta+\gamma)=0$

$$\tan\alpha+\tan\beta+\tan\gamma=\tan(\alpha+\beta)\cdot(1-\tan\alpha\cdot\tan\beta)+\tan\gamma$$
$$\tan\alpha+\tan\beta+\tan\gamma=\tan(\alpha+\beta)+\tan\gamma-\tan\alpha\cdot\tan\beta\cdot\tan(\alpha+\beta)$$
$$\tan\alpha+\tan\beta+\tan\gamma=\tan(\alpha+\beta+\gamma)\cdot(1-\tan\alpha\cdot\tan\beta\cdot\tan\gamma)+\tan\alpha\cdot\tan\beta\cdot\tan\gamma$$

Hence, $\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\cdot\tan\beta\cdot\tan\gamma$ if $\alpha+\beta+\gamma=180°$

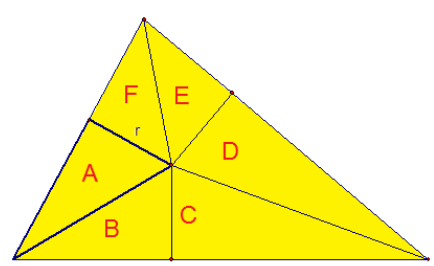
Proof:



AB, BC and AC are tangents to the circle.
Tangents from an external point are equal in length, therefore, the sides of the triangle can be divided using x, y and z as shown above, where
a = x + y, b = y + z, c = x + z

Let $s=\dfrac{a+b+c}{2}$,

$$s=\frac{(x+y)+(y+z)+(x+z)}{2}=x+y+z$$
$$s-a=(x+y+z)-(x+y)=z$$
$$s-b=(x+y+z)-(z+y)=x$$
$$s-c=(x+y+z)-(x+z)=y$$
$$\therefore (s-a)\cdot(s-b)\cdot(s-c)=xyz$$



The angles at the centre formed by perpendiculars to the sides are α, β and γ as shown above

In the triangle A,

$$\tan\frac{\alpha}{2}=\frac{x}{r}\Leftrightarrow x=r\tan\frac{\alpha}{2}$$

Similarly by considering triangle C and E,

$y=r\tan\frac{\beta}{2}$ and $z=r\tan\frac{\gamma}{2}$

$$x + y + z = r\left(\tan \alpha/2 + \tan \beta/2 + \tan \gamma/2\right)$$

$$x + y + z = r\left(\tan \alpha/2 \cdot \tan \beta/2 \cdot \tan \gamma/2\right) \cdots \left(\alpha/2 + \beta/2 + \gamma/2 = 180^\circ\right)$$

$$x + y + z = r\left(\frac{x}{r} \cdot \frac{y}{r} \cdot \frac{z}{r}\right)$$

$$r^2(x + y + z) = xyz$$

$$\therefore r^2 = \frac{xyz}{(x + y + z)}$$

$$r = \sqrt{\frac{xyz}{(x + y + z)}}$$

$$r = \sqrt{\frac{(s - a) \cdot (s - b) \cdot (s - c)}{s}}$$

By Tran Minh Tri

Extension to Heron formulae:
Area of triangle:

$$\begin{aligned} Area &= r \times x + r \times y + r \times z \\ &= r(x + y + z) \\ &= s\sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \\ &= \sqrt{s(s-a)(s-b)(s-c)} \end{aligned}$$

Added on 30/10/2009