The following question was set in 'A' level in 2000. This can now appear in both IB HL Math and 'A' level H2 math examination.

We can solve this by 3 different methods, namely Mathematical Induction, Mathematical Induction with the help of ratio theorem and Method of Difference.

All the terms of the AP $u_1, u_2, u_3, \dots, u_n$, are positive. Prove that, for $n \geq 2$, $\sum_{k=2}^{n} \frac{1}{u_k u_{k-1}} = \frac{n-1}{u_1 u_n}$.

Method 1:

Let P(n) be the statement $\sum_{k=2}^{n} \frac{1}{u_k u_{k-1}} = \frac{n-1}{u_1 u_n}$ for $n \geq 2$.

LHS of P(2) = $\frac{1}{u_1 u_2} = \frac{2-1}{u_1 u_2}$ =RHS of P(2), hence P(2) is true.

Assume P(m) if true for some $m \geq 2$, i.e.

$$\sum_{k=2}^{m} \frac{1}{u_k u_{k-1}} = \frac{m-1}{u_1 u_m},$$

We would like to show that P(m+1) is also true.

LHS of P(m+1) $= \frac{m-1}{u_1 u_m} + \frac{1}{u_{m+1} u_m} = \frac{(m-1)u_{m+1} + u_1}{u_m(u_1 u_{m+1})}$

$$= \frac{m u_{m+1} - (u_{m+1} - u_1)}{u_m(u_1 u_{m+1})}$$

$$= \frac{m[u_{m+1} - (u_{m+1} - u_m)]}{u_m(u_1 u_{m+1})}$$, d is common difference

$$= \frac{m}{u_1 u_{m+1}}$$ = RHS of P(m+1)

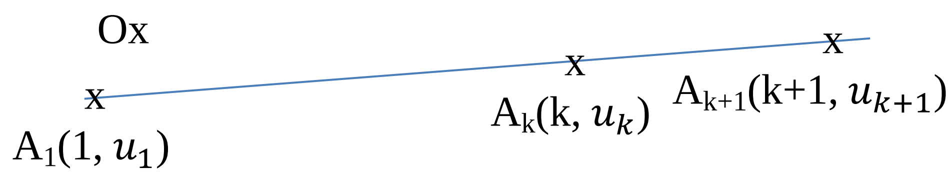
Hence P(m+1) is also true.

By Mathematical induction, P(n) is true for $n \geq 2$.

Method 2:

LHS of $P_{k+1} = \frac{k-1}{u_1 u_k} + \frac{1}{u_k u_{k+1}} = \frac{(k-1)u_{k+1} + u_1}{u_1 u_k u_{k+1}}$ (**)

Since u_k forms an AP, then clearly the points $A_1(1, u_1)$, $A_2(2, u_2)$,.., $A_k(k, u_k)$, $A_{k+1}(k+1, u_{k+1})$ will form a straight line,



Note that the ratio $A_1A_k : A_kA_{k+1} = (k-1): 1$

Hence by applying the ratio theorem, $\mathbf{OA_k} = \frac{\mathbf{OA_1} + (\boldsymbol{k}-\mathbf{1})\mathbf{OA_{k+1}}}{\mathbf{1} + (\boldsymbol{k}-\mathbf{1})}$

i.e. $\begin{pmatrix} k \\ u_k \end{pmatrix} = \frac{\begin{pmatrix} 1 \\ u_1 \end{pmatrix} + (k-1)\begin{pmatrix} k+1 \\ u_{k+1} \end{pmatrix}}{k} = \frac{1}{k}\begin{pmatrix} k^2 \\ (k-1)u_{k+1} + u_1 \end{pmatrix}$

i.e. $u_k = \frac{1}{k}[(k-1)u_{k+1} + u_1]$

i.e. $(k-1)u_{k+1} + u_1 = k u_k$

Hence, (**)= $\frac{k u_k}{u_1 u_k u_{k+1}} = \frac{k}{u_1 u_{k+1}}$ =RHS of P_{k+1}

Method 3: (note that this is NOT in IB HL Math syllabus)

Using partial fraction: $\frac{1}{u_k u_{k-1}} = \frac{1}{d}\left(\frac{1}{u_{k-1}} - \frac{1}{u_k}\right)$

$$\sum_{k=2}^{n} \frac{1}{u_k u_{k-1}} = \frac{1}{d}\sum_{k=2}^{n}\left(\frac{1}{u_{k-1}} - \frac{1}{u_k}\right) = \frac{1}{d}\left(\frac{1}{u_1} - \frac{1}{u_n}\right) = \frac{u_n - u_1}{d(u_1 u_n)} = \frac{n-1}{u_1 u_n}.$$

Extra Q:

Given that $u_1, u_2, u_3, \dots, u_n$ are the 1st n terms of a GP. Prove that

$$\left(\sum_{k=1}^{n} u_k\right) : \left(\sum_{k=1}^{n} \frac{1}{u_k}\right) = u_1 u_n : 1$$