Q) Why can we perform $h^{-1}hg(x)$ to find g(x) and not $hgg^{-1}(x)$ to find h(x)?

Ans:

eg. if $g(x)=x+1$ for $3<x<4$, then $g^{-1}(x)=x-1$ for $4<x<5$ i,e, $gg^{-1}(x)=x$ for $4<x<5$. but if h is only defined outside $4<x<5$, then $hgg^{-1}(x)$ does not exist as a function.

Similarly $h^{-1}h(x)=i(x)=x$ if x is an element of d_h. if hg(x) is given (i.e. Q assume hg(x) exists a function) i.e. r_g is a subset of d_h. by performing $h^{-1}hg(x)=ig(x)$ where domain of i(x)=x is the same as domain of h(x). As such you won't get any contradicting conclusion.

So i also agree that $h^{-1}hg(x)$ to find g(x) is ok in general but not $hgg^{-1}(x)$ to find h(x).